# PLANE HARMONIC WAVES IN AN INFINITELY LARGE THERMOELASTIC MEDIUM WITH A FINITE VELOCITY OF HEAT PROPAGATION

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The properties of coupled thermoelastic harmonic waves are analyzed, with the finite velocity of heat propagation taken into account.

Let a plane wave travel in an infinitely large medium along the $x_1$-axis, this wave being a harmonic function of time. Such a wave may be excited thermally or mechanically. At any instant of time such a plane wave is fixed in any plane orthogonal to the direction of travel $x_1$ and the temperature is constant. For this reason, the displacements $u_i$ and the temperature $\theta$ are functions of the space coordinate $x_1$ and of time t. In this case the equations of coupled thermoelasticity with a finite velocity of heat propagation become much simpler:

$$\left(\frac{\partial^2}{\partial x_1^2}-\frac{1}{C_1^2}\frac{\partial^2}{\partial t^2}\right)u_1=m\frac{\partial\theta}{\partial x_1},\quad \left(\frac{\partial^2}{\partial x_1^2}-\frac{1}{C_2^2}\frac{\partial^2}{\partial t^2}\right)u_2=0,$$
$$\left(\frac{\partial^2}{\partial x_1^2}-\frac{1}{C_2^2}\frac{\partial^2}{\partial t^2}\right)u_3=0, \tag{1}$$

$$\left(\frac{\partial^2}{\partial x_1^2}-\frac{1}{a}\frac{\partial}{\partial t}-\frac{1}{v_T^2}\frac{\partial^2}{\partial t^2}\right)\theta-\eta\left(\frac{\partial}{\partial t}+\tau^*\frac{\partial^2}{\partial t^2}\right)\frac{\partial u_1}{\partial x_1}=0, \tag{2}$$

where

$$m=\frac{\alpha_T(3\lambda+2\mu)}{\lambda+2\mu};\quad \eta=\frac{T_0\alpha_T(3\lambda+2\mu)}{C_e\varkappa}.$$

All quantities vary harmonically and, therefore,

$$u_i=\mathrm{Re}\left[u_i^*(x_i,\ \omega)\,e^{-i\omega t}\right],$$
$$\theta=\mathrm{Re}\left[\theta^*(x_i,\ \omega)\,e^{-i\omega t}\right]. \tag{3}$$

Inserting (3) into (1) and (2), we have

$$\left(\frac{\partial^2}{\partial x_1^2}+n^2\right)u_1^*=m\frac{\partial\theta^*}{\partial x_1},$$
$$\left(\frac{\partial^2}{\partial x_1^2}+q+P^2\right)\theta^*+\eta a(q+P^2)\frac{\partial u_1^*}{\partial x_1}=0, \tag{4}$$
$$\left(\frac{\partial^2}{\partial x_1^2}+n_1^2\right)u_2^*=0,$$

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$$\left(\frac{\partial^2}{\partial x_1^2}+n_1^2\right)u_3^*=0, \tag{5}$$

where

$$n^2=\frac{\omega^2}{C_1^2};\quad n_1^2=\frac{\omega^2}{C_2^2};\quad q=\frac{i\omega}{a};\quad P^2=\frac{\omega^2}{v_T^2}. \tag{6}$$

Since transverse waves at a fixed frequency $\omega$-travel at a constant velocity, without causing volume changes or distortions of the temperature field, let us analyze the system of equations (4).

We eliminate from (4) the temperature $\theta$ and seek the solution to the resulting equation in the form:

$$u_1^x=u_0e^{ikx_1},\quad \theta_1^x=\theta_0e^{ikx_1}, \tag{7}$$

which yields the characteristic equation

$$k^4-k^2\left[n^2+(q+P^2)(1+\varepsilon)\right]+n^2(q+P^2)=0, \tag{8}$$

where $\varepsilon = m\eta a$ is the coupling parameter.

We now calculate the roots of Eq. (8):

$$k_{1,2}^2=\frac{1}{2}\left\{n^2+(q+P^2)(1+\varepsilon)\pm\sqrt{[n^2+(q+P^2)(1+\varepsilon)]^2-4n^2(q+P^2)}\right\}. \tag{9}$$

When $\varepsilon = 0$, we have

$$k_1=n,\quad k_2=\sqrt{q+P^2}. \tag{10}$$

Denoting the phase velocity by $V_\beta$ ($\beta = 1, 2$) and the attenuation factor by $\vartheta_\beta$, we have

$$V_\beta=\frac{\omega}{\mathrm{Re}\,(k_\beta)},\quad \vartheta_\beta=\mathrm{Im}\,(k_\beta). \tag{11}$$

Inserting (7) into (4) yields

$$\frac{n^2-k^2}{mik}=-\frac{\eta aik(q+P^2)}{q+P^2-k^2},\qquad \frac{\theta_0}{u_0}=-\frac{\eta aik(q+P^2)}{q+P^2-k^2}. \tag{12}$$

Now we can write the solution to system (4) as

$$u_1=u_+^0\exp[-i\omega t+ik_1x_1]+u_-^0\exp[-i\omega t-ik_1x_1]$$
$$+\frac{mik_2}{n^2-k_2^2}\{\theta_+^0\exp[-i\omega t+ik_2x_1]-\theta_-^0\exp[-i\omega t-ik_2x_1]\}, \tag{13}$$

$$\theta=\theta_+^0\exp[-i\omega t+ik_2x_1]+\theta_-^0\exp[-i\omega t-ik_2x_1]$$
$$+\frac{\eta aik_1(q+P^2)}{k_1^2-m^2-q}\{u_+^0\exp[-i\omega t+ik_1x_1]+u_-^0\exp[-i\omega t-ik_1x_1]\}. \tag{14}$$

We have obtained here expressions for longitudinal thermoelastic waves traveling at a constant frequency along axes $x_1$ and $-x_1$.

Taking into account (11), we rewrite (13) and (14) as follows:

$$u_1=u_+^0\exp\left[-i\omega\left(t-\frac{x_1}{V_1}\right)-\vartheta_1x_1\right]+u_-^0\exp\left[-i\omega\left(t+\frac{x_1}{V_1}\right)+\vartheta_1x_1\right]+\frac{mik_2}{n^2-k_2^2}$$
$$\times\left\{\theta_+^0\exp\left[-i\omega\left(t-\frac{x_1}{V_2}\right)-\vartheta_2x_1\right]-\theta_-^0\exp\left[-i\omega\left(t+\frac{x_1}{V_2}\right)+\vartheta_2x_1\right]\right\}, \tag{15}$$

$$\theta=\theta_+^0\exp\left[-i\omega\left(t-\frac{x_1}{V_2}\right)-\vartheta_2x_1\right]+\theta_-^0\exp\left[-i\omega\left(t+\frac{x_1}{V_2}\right)+\vartheta_2x_1\right]$$
$$+\frac{\eta aik_1(q+P^2)}{k_1^2-m-q}\left\{u_+^0\exp\left[-i\omega\left(t-\frac{x_1}{V_1}\right)-\vartheta_1x_1\right]-u_-^0\exp\left[-i\omega\left(t+\frac{x_1}{V_1}\right)+\vartheta_1x_1\right]\right\}. \tag{16}$$

Formulas (15) and (16) describe a modified elastic wave and a heat wave, respectively, these waves being subject to attenuation and dispersion, while without coupling between the temperature field and the strain field taken into account there would be neither attenuation nor dispersion.

We now transform (9) after introducing the following notation

$$\psi = \frac{C_1}{\omega^x} k, \quad \omega^x = \frac{C_1^2}{a}, \quad \chi = \frac{\omega}{\omega^x}, \tag{17}$$

so that

$$\psi^4 - \psi^2\chi\left[\chi + (1+\varepsilon)(i + M^2\chi)\right] + \chi^3(i + \chi M^2) = 0, \tag{18}$$

with $M = C_1/V_T$.

The solution to (18) is

$$\psi_{1,2}^2 = \frac{\chi}{2}\{\chi + (1+\varepsilon)(i+\chi M^2) \pm \sqrt{\chi^2 - 2(1-\varepsilon)\chi(i+M^2\chi) + (1+\varepsilon)^2(i+\chi M^2)^2}\} \tag{19}$$

or

$$\psi_1 = \sqrt{\frac{\chi}{2}}\{\chi + (1+\varepsilon)\chi M^2 + i(1+\varepsilon) + \left[\chi^2 - 2i\chi(1-\varepsilon) + (1+\varepsilon)^2(i+\chi M^2)^2\right]^{1/2}\}^{1/2}, \tag{20}$$

$$\psi_2 = \sqrt{\frac{\chi}{2}}\{\chi + (1+\varepsilon)\chi M^2 + i(1+\varepsilon) - \left[\chi^2 - 2i\chi(1-\varepsilon) + (1+\varepsilon)^2(i+\chi M^2)^2\right]^{1/2}\}^{1/2}. \tag{21}$$

The roots depend on the natural frequency of the material $\omega^x = C_1^2/a$, the coupling parameter $\varepsilon$, the Mach number M, and the parameter $\chi = \omega/\omega^x$. For feasible mechanical vibrations we may assume that $\chi \ll 1$.

We next expand (20) into a power series in $\chi$ and change to $k_1$:

$$k_1 = \frac{\omega}{C_1(1+\varepsilon)^{1/2}}\left\{1 + \frac{\chi^2}{8(1+\varepsilon)^4}\left[\varepsilon(4-3\varepsilon) + M^2(1+\varepsilon)^2\left(2\varepsilon(3-\varepsilon)\right.\right.\right.$$

$$\left.\left.\left. - M^2(2+\varepsilon)(1-\varepsilon^2) + M^4(1+\varepsilon)^4\right)\right] + \frac{i\varepsilon\chi}{2(1+\varepsilon)^2}\right\}. \tag{22}$$

Separating the real part and the imaginary part of (22), we have

$$\mathrm{Re}\, k_1 = \frac{\omega}{C_s}\left\{1 + \frac{\chi^2}{8(1+\varepsilon)^4}\left[\varepsilon(4-3\varepsilon) + M^2(1+\varepsilon)^2\left(2\varepsilon(3-\varepsilon) - M^2(2+\varepsilon)(1-\varepsilon^2) + M^4(1+\varepsilon)^4\right)\right]\right\}, \tag{23}$$

$$\mathrm{Im}\, k_1 = \frac{\varepsilon\omega\chi}{2C_s(1+\varepsilon)^2}, \tag{24}$$

where $C_s = C_1(1+\varepsilon)^{1/2}$ is the adiabatic phase velocity of the expansion wave.

According to (11), we define the phase velocity as follows:

$$V_1 = \frac{\omega}{\mathrm{Re}\, k_1} = C_s\left\{1 - \frac{\chi^2}{8(1+\varepsilon)^4}\left[\varepsilon(4-3\varepsilon) + M^2(1+\varepsilon)^2\left(2\varepsilon(3-\varepsilon) - M^2(2+\varepsilon)(1-\varepsilon^2) + M^4(1+\varepsilon)^4\right)\right]\right\}. \tag{25}$$

When $\chi \ll 1$, therefore, the phase velocity is independent of the frequency $\omega$, i.e., the expansion wave has no dispersion, but depends on the velocity of heat propagation instead.

Formula (25) can be used for determining the velocity of heat propagation from tests, which requires an accurate measurement of the phase velocity of the longitudinal elastic wave, inasmuch as $C_s = C_1(1+\varepsilon)^{1/2}$, $C_1^2 = (\lambda + 2\mu)/\rho$, etc. From (25) one can calculate $M = C_1/V_T$ for $\chi \ll 1$. Coefficient $\mathrm{Im}\, k_1 = \vartheta_1$ is a positive quantity and, therefore, the amplitude of the wave decreases exponentially within the heat conducting medium.

Analogously we can determine the phase velocity $V_2$ and the attenuation factor $\vartheta_2$ of a quasithermal wave.

When the velocity of heat propagation is infinite, then $M = 0$ and

$$V_1 = C_s\left(1 - \frac{\varepsilon(4-3\varepsilon)}{8(1+\varepsilon)^4}\chi^2\right) \approx C_s. \qquad (26)$$

In this particular case one may assume that, when M > 1, the $M^6(1+\varepsilon)^6$ term contributes most to the phase velocity and, for this reason, one may rewrite (25) as

$$V_1 = C_s\left\{1 - \frac{\varepsilon(4-3\varepsilon) + M^6(1+\varepsilon)^6}{8(1+\varepsilon)^4}\chi^2\right\}, \qquad (27)$$

or

$$V_1 = C_s\left\{1 - \frac{\varepsilon(4-3\varepsilon) + 2M^2\varepsilon(3-\varepsilon)(1+\varepsilon)^2}{8(1+\varepsilon)^4}\chi^2\right\}. \qquad (28)$$

when M < 1.

For the wave considered here we will determine now the relative energy dispersion $\Delta W/W$, where $\Delta W$ denotes the energy dispersed during one stress cycle and W denotes the elastic energy stored in the body up to the instant of time when the strain becomes maximum. Let $u_1$ and $u_2$ be successive displacement amplitudes. Approximately, $\Delta W/W$ can be expressed as

$$\frac{\Delta W}{W} = \frac{u_1^2 - u_2^2}{u_1^2} \simeq \frac{2(u_1 - u_2)}{u_2} = 2\ln\frac{u_1}{u_2}. \qquad (29)$$

However,

$$u_1 = u_2 \exp\left[\frac{2\Pi}{\mathrm{Re}\,k_1}\,\mathrm{Im}\,k_1\right],$$

and taking into account (23)-(24), we obtain

$$\frac{\Delta W}{W} = \frac{4\Pi\,\mathrm{Im}\,k_1}{\mathrm{Re}\,k_1} = \frac{2\Pi\varepsilon\chi}{(1+\varepsilon)^2}\left\{1 - \frac{\chi^2}{8(1+\varepsilon)^4}\left[\varepsilon(4-3\varepsilon) + M^2(1+\varepsilon)^2\left(2\varepsilon(3-\varepsilon) - M^2(2+\varepsilon)(1-\varepsilon^2)\right) + M^4(1+\varepsilon)^4\right]\right\} \qquad (30)$$

or

$$\frac{\Delta W}{W} = \frac{2\Pi\varepsilon\chi}{(1+\varepsilon)^2}\left\{1 - \frac{\chi^2}{8(1+\varepsilon)^4}\left[\varepsilon(4-3\varepsilon) + M^6(1+\varepsilon)^6\right]\right\}. \qquad (31)$$

The phase velocity of a thermoelastic wave is equal to the adiabatic phase velocity of the expansion wave when the velocity of heat propagation is infinite [1, 3], but, according to Eqs. (26) and (27), is lower than that by an amount of the order of about $(M^6\lambda^2/8)C_S$ when the velocity of heat propagation is recognized as finite. When the relaxation time in solids, as shown in [2] for conductors at M > 1, is accounted for, then obviously the decrease in the phase velocity ceases to be negligible. In polymers at M = 1 the decrease in the phase velocity is insignificant, even when the velocity of heat propagation is properly considered as finite. An analysis of the relative energy dispersion according to (31) indicates that, when the velocity of heat propagation is assumed infinite (M = 0), it is higher than the relative energy dispersion at a finite velocity of heat propagation.

Thus, the preceding analysis of coupled harmonic thermoelastic waves has shown that not accounting for the finite velocity of heat propagation will yield a higher value for the phase velocity of thermoelastic waves in metals and also in a higher value for the relative energy dispersion. The result is analogous for thermoelastic waves traveling in polymers, but the discussed effects are weaker. In polymers with a coupling parameter $\varepsilon$ within the 0.2-0.5 range these effects may not be disregarded.

## NOTATION

u is the displacement;
$\theta$ is the temperature;
$C_1$ is the longitudinal velocity of sound in the medium;
$C_2$ is the transverse velocity of perturbations in the medium;
$\omega$ is the frequency;

$\varkappa$ is the thermal conductivity;
$V_T$ is the velocity of heat propagation;
$\lambda, \mu$ are the Lamé constants;
$a$ is the thermal diffusivity;
$\tau^*$ is the relaxation time.

## LITERATURE CITED


1. A. D. Kovalenko, Fundamentals of Thermoelasticity [in Russian], Naukova Dumka (1970).
2. A. V. Lykov, Theory of Heat Conduction [in Russian], Vysshaya Shkola (1967).
3. W. Nowacki, Problems in Dynamic Thermoelasticity [Russian translation], Mir, Moscow (1970).